Student seminar exercise sheet Week 6

1. Let H be a subgroup of $\mathbb{Z}/n\mathbb{Z}$ and define
$$\mathcal{S} = \{\, p \text{ prime in } \mathbb{Z} : p + n\mathbb{Z} \in H \,\}.$$
Compute the density $\delta(\mathcal{S})$ in terms of $\#H$, and prove that your result is correct.

2. Let K/F be Galois, and let
$$\mathcal{S}_{K/F} = \{\mathfrak{p} \in \mathcal{O}_F : \mathfrak{p} \text{ splits completely in } K/F\}.$$
Prove that
$$\delta_F(\mathcal{S}_{K/F}) = \frac{1}{[K:F]}.$$

3. Let E and K be number fields, each of which is Galois over $\mathbb{Q}$.
Show that for any prime $p \in \mathbb{Z}$, p splits completely in both $E/\mathbb{Q}$ and $K/\mathbb{Q}$ if and only if p splits in their compositum extension $KE/\mathbb{Q}$.

4. Let F be a number field and $\mathfrak{m} \subset \mathcal{O}_F$ a non-zero ideal. Let $P^+_{F,\mathfrak{m}}$ be the subgroup of P_F generated by
$$\{\langle\alpha\rangle : \alpha \in \mathcal{O}_F,\ \alpha \equiv 1 \pmod{\mathfrak{m}},\ \text{and } \alpha \gg 0\}.$$
Show the two equalities:
$$P^+_{F,\mathfrak{m}} = \{\langle\alpha\rangle :\ \alpha \in F^\times,\ \alpha \gg 0,\ \alpha \overset{\times}{\equiv} 1 \pmod{\mathfrak{m}}\}$$
$$= \{\langle\tfrac{\alpha}{\beta}\rangle :\ \tfrac{\alpha}{\beta} \gg 0,\ \alpha, \beta \in \mathcal{O}_F \text{ prime to } \mathfrak{m},\ \alpha \equiv \beta \pmod{\mathfrak{m}}\}.$$